

“When the rules of quantum mechanics were formulated in the 1920’s they represented a revolutionary break with the past, and an enormous extrapolation from experience. Since they were something very new, they could not be derived from something old and incorrect, that is, classical physics. Instead they had to be formulated by guessing, intuition, and inspiration. Their ultimate justification was, and is, logical consistency and agreement with experiment.”

- Prof. Eugene D. Commins, U.C. Berkeley.

If you have any questions, suggestions or corrections to the solutions, don’t hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

(a)

The uncertainty principle yields an estimate for the minimum momentum of a proton trapped in the nucleus:

$$\Delta p \approx \frac{\hbar}{2\Delta r}. \quad (1)$$

For the kinetic energy of the proton, we obtain:

$$E_k = \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{8m(\Delta r)^2} = \frac{\hbar^2 c^2}{8mc^2(\Delta r)^2} = \frac{197.3 \text{ MeV} \cdot \text{fm}}{8(938 \text{ MeV})(2 \text{ fm})^2}, \quad (2)$$

from which we find

$$E_k = 1.4 \text{ MeV}$$

(b)

We’ll suppose, for the sake of this “back of the envelope” calculation, that the kinetic energy found in part (a) can be set equal to the potential energy at maximum displacement in a classical harmonic oscillator:

$$E_k = \frac{1}{2} k x_0^2 \quad (3)$$

The magnitude of the restoring force at maximum displacement is given by kx_0 . So we find:

$$F = \frac{2E_k}{x_0} = 1.4 \text{ MeV/fm}.$$

The strength of the electric force is given by

$$F_e = \frac{ke^2}{x_0^2} = \frac{1.44 \text{ MeV} \cdot \text{fm}}{(2 \text{ fm})^2} = 0.36 \text{ MeV/fm}.$$

(c)

The maximum acceleration of the proton is given by:

$$a_{\text{max}} = \frac{F}{m} = \frac{kx_0 c^2}{mc^2} = 1.3 \times 10^{29} \text{ m/s}^2 \sim 10^{28} g. \quad (4)$$

Problem 2

It is convenient to write the differential cross section as $\frac{d\sigma}{d\cos\theta}$ instead of $\frac{d\sigma}{d\theta}$ because it makes integration over solid angles a little easier, since the integral always involves $\cos\theta$ and the differential solid angle contains the term $\sin\theta d\theta = -d(\cos\theta)$.

It is relatively straightforward to show that either method of solving for the total cross section gives the same result, since

$$\frac{d\sigma}{d\theta} = \frac{d\sigma}{d\cos\theta} \cdot \frac{d\cos\theta}{d\theta} = -\sin\theta \frac{d\sigma}{d\cos\theta}.$$

If we integrate over all angles, we obtain for the total cross section:

$$\sigma = \int_{-1}^{+1} d(\cos\theta) \frac{d\sigma}{d\cos\theta} = \int_0^\pi d\theta \sin\theta \frac{1}{\sin\theta} \frac{d\sigma}{d\theta}.$$

Problem 3

(a)

The maximum kinetic energy that can be transferred to a gold nucleus in a collision with a 6 MeV α -particle would be when the collision is head-on and the α -particle bounces straight back. Because the gold nucleus is very massive compared to the α -particle, the amount of kinetic energy transferred to the gold nucleus should be small, so roughly $v_i = -v_f$ where v_i and v_f are the initial and final velocities of the α -particle. Thus, $M_{\text{gold}} V = 2m_\alpha v_i$. Using this result in the equation for kinetic energy, we find:

$$\frac{1}{2} M_{\text{gold}} V^2 = \frac{1}{2} M_{\text{gold}} \left(\frac{2m_\alpha v_i}{M_{\text{gold}}} \right)^2 = \left(\frac{4m_\alpha}{M_{\text{gold}}} \right) \left(\frac{1}{2} m_\alpha v_i^2 \right) = \frac{4 \cdot 4}{197} \cdot 6 \text{ MeV} \approx 0.49 \text{ MeV}$$

(b)

If we go into the rest frame of the α -particle (\mathcal{S}'), we find that because the α -particle is very massive compared to the electron, the energy transferred to the α -particle (to the electron in the lab frame) is small. Therefore we can approximate that in \mathcal{S}' , for a head-on collision $m_\alpha V' = 2m_e v_i$, where v_i is the speed of the α -particle in the lab frame and V' is the recoil speed of the α -particle in \mathcal{S}' . Transforming into the lab frame, we find that the recoil speed of the electron $v_e = 2v_i$. So the kinetic energy transferred to the electron is

$$E_k = \frac{1}{2}m_e(2v_i)^2 = \frac{4m_e}{m_\alpha} \left(\frac{m_\alpha v_i^2}{2} \right) = \frac{4 \cdot (0.511 \text{ MeV})}{3730 \text{ MeV}} 6 \text{ MeV} \approx 3.3 \text{ keV}.$$

Problem 4

(a)

The relationship between differential scattering cross section $d\sigma$ and the impact parameter b is given by Rohlf (6.18):

$$d\sigma = 2\pi b db \quad (5)$$

The total scattering cross section is derived from this expression:

$$\sigma = 2\pi \int_{b_2}^{b_1} b db = \pi(b_1^2 - b_2^2). \quad (6)$$

Using Rohlf (6.40)

$$\left(\frac{kq_1 q_2}{mv^2} \right)^2 \left(\frac{1 + \cos \theta}{1 - \cos \theta} \right) = b^2$$

we find that the total scattering cross section is given by:

$$\sigma = 2\pi \left(\frac{kq_1 q_2}{mv^2} \right)^2 \left(\frac{\cos \theta_1 - \cos \theta_2}{(1 - \cos \theta_1)(1 - \cos \theta_2)} \right). \quad (7)$$

(b)

Integrating explicitly gives us the same result:

$$\sigma = \int_{\cos \theta_2}^{\cos \theta_1} d(\cos \theta) \frac{d\sigma}{d \cos \theta}$$

$$\begin{aligned} \sigma &= 2\pi \left(\frac{kq_1 q_2}{mv^2} \right)^2 \int_{\cos \theta_2}^{\cos \theta_1} d(\cos \theta) \frac{1}{(1 - \cos \theta)^2} \\ \sigma &= 2\pi \left(\frac{kq_1 q_2}{mv^2} \right)^2 \left(\frac{\cos \theta_1 - \cos \theta_2}{(1 - \cos \theta_1)(1 - \cos \theta_2)} \right) \end{aligned}$$

Problem 5

A particle is confined to the region $-L/2 < x < L/2$. As discussed in section, this means that any state (wavefunction) of the particle can be described as a superposition of eigenfunctions of the energy operator (the Hamiltonian). These eigenfunctions “span” the Hilbert space corresponding to our system (a Hilbert space is an infinite dimensional vector space which is a subspace of the vector space of all continuous complex functions). Since inside the infinite potential well the particle is free, our Hamiltonian \mathcal{H} is given by:

$$\mathcal{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}.$$

Eigenfunctions of \mathcal{H} are

$$\psi_n = \sqrt{\frac{2}{L}} \cos \left(\frac{n\pi x}{L} \right)$$

$$\psi_m = \sqrt{\frac{2}{L}} \sin \left(\frac{m\pi x}{L} \right)$$

where $n = 1, 3, 5, \dots$ and $m = 2, 4, 6, \dots$. They have the eigenvalues

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

where $n = 1, 2, 3, \dots$. These eigenfunctions are orthonormal, meaning that

$$\int_{-L/2}^{L/2} \psi_i^*(x) \psi_j(x) dx = 0$$

if $i \neq j$, and

$$\int_{-L/2}^{L/2} \psi_i^*(x) \psi_i(x) dx = 1.$$

(a)

Assume the particle is in an eigenstate of energy. The probability that the particle is found in the region $0 < x < L/2$ is $1/2$ by symmetry. This is because

the potential is symmetric about $x = 0$, so every eigenfunction is symmetric or antisymmetric about $x = 0$. The square of any eigenfunction is symmetric about $x = 0$.

It is clear that probability does not depend on n because all of the eigenfunctions are symmetric or antisymmetric.

(b)

The probability P_c that a particle in the ground state is in the central half of the box is given by the integral:

$$P_c \int_{-L/4}^{L/4} dx |\psi_1|^2 = \frac{2}{L} \int_{-L/4}^{L/4} dx \cos^2(\pi x/L). \quad (8)$$

From which we find:

$$P_c = \frac{2}{L} \left[\frac{x}{2} + \frac{\sin(2\pi x/L)}{(4\pi/L)} \right]_{x=-L/4}^{x=L/4} = 0.82.$$

The probability decreases with n , and at large n approaches the classical limit $P_c = 0.5$.

Problem 6

(a)

The average value (expectation value) of x^2 as a function of n is given by:

$$\langle x^2 \rangle = \int_{-L/2}^{L/2} dx \psi_n^* \cdot x^2 \cdot \psi_n. \quad (9)$$

We use the eigenfunctions discussed in problem (5), solving first for even n . In this case $\langle x^2 \rangle$ is given by:

$$\langle x^2 \rangle = \frac{2}{L} \int_{-L/2}^{L/2} x^2 \sin^2(n\pi x/L) dx.$$

Making the u -substitution $u = n\pi x/L$, we obtain:

$$\langle x^2 \rangle = \frac{2L^2}{m^3\pi^3} \int_{-m\pi/2}^{m\pi/2} u^2 \sin^2 u du$$

$$\langle x^2 \rangle = \frac{2L^2}{m^3\pi^3} \left[\frac{u^3}{6} - \frac{u^2 \sin(2u)}{4} + \frac{\sin(2u)}{8} - \frac{4 \cos(2u)}{4} \right]_{-m\pi/2}^{m\pi/2}.$$

$$\langle x^2 \rangle = \frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}.$$

For odd n , the procedure is pretty much the same... you even end up with the same result.

$$\langle x^2 \rangle = \frac{2}{L} \int_{-L/2}^{L/2} x^2 \cos^2(n\pi x/L) dx.$$

Making the u -substitution $u = n\pi x/L$, we obtain:

$$\langle x^2 \rangle = \frac{2L^2}{m^3\pi^3} \int_{-m\pi/2}^{m\pi/2} u^2 \cos^2 u du$$

$$\langle x^2 \rangle = \frac{2L^2}{m^3\pi^3} \left[\frac{u^3}{6} + \frac{u^2 \sin(2u)}{4} - \frac{\sin(2u)}{8} - \frac{4 \cos(2u)}{4} \right]_{-m\pi/2}^{m\pi/2}.$$

$$\langle x^2 \rangle = \frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}.$$

Taking the limit as $n \rightarrow \infty$, we see that the rms value of x approaches $L/\sqrt{12}$.

Problem 7

We intend to prove the conservation of probability law:

$$\frac{\partial}{\partial t}(\psi^* \psi) + \frac{\partial}{\partial x} j = 0, \quad \text{where}$$

$$\frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right) \equiv j,$$

and j is the *probability current* in one dimension x , where $\psi(x, t)$ is a solution of the Schrödinger equation with a real potential $V(x)$.

We can start with the time-dependent Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t). \quad (10)$$

Then take the complex conjugate of (10):

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi^*(x, t) = -i\hbar \frac{\partial}{\partial t} \psi^*(x, t). \quad (11)$$

Now multiply (10) by $\psi^*(x, t)$ and (11) by $\psi(x, t)$, then subtract the equations. We obtain:

$$\psi^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi - \psi \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi^* = i\hbar \left(\psi^* \frac{\partial}{\partial t} \psi + \psi \frac{\partial}{\partial t} \psi^* \right). \quad (12)$$

From which we can deduce:

$$-\frac{\hbar}{2mi} \left(\psi^* \frac{\partial^2}{\partial x^2} \psi - \psi \frac{\partial^2}{\partial x^2} \psi^* \right) = \frac{\partial(\psi\psi^*)}{\partial t}. \quad (13)$$

Now consider $\frac{\partial}{\partial x} j$:

$$\frac{\partial}{\partial x} j = \frac{\partial}{\partial x} \left(\frac{\hbar}{2mi} \right) \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) \quad (14)$$

$$\frac{\partial}{\partial x} j = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial^2}{\partial x^2} \psi - \psi \frac{\partial^2}{\partial x^2} \psi^* \right) \quad (15)$$

If we use Eq. (15) in Eq. (13), we obtain the conservation of probability law:

$$\frac{\partial}{\partial t} (\psi^* \psi) + \frac{\partial}{\partial x} j = 0. \quad (16)$$

Problem 8

There was a correction to this problem, specifically that the initial wavefunction of the particle in the box is supposed to be

$$u(x) \propto \sin(\pi x/L) + \sin(2\pi x/L)$$

instead of

$$u'(x) \propto \exp(i\pi x/L) + \exp(i2\pi x/L).$$

It is useful to consider the problem with $u'(x)$. If we solve for the energy eigenfunctions of the Hamiltonian for this problem

$$\mathcal{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2},$$

we obtain:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad (17)$$

which correspond to the energy eigenvalues:

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}. \quad (18)$$

From the postulates of quantum mechanics (discussed in section), we know that any state of an isolated system corresponds to a function in the corresponding Hilbert space. This Hilbert space is spanned by the eigenfunctions of a Hermitian operator (which corresponds to an observable, in this case energy). Therefore, if $u'(x)$ were a state in our system, it could be represented as a superposition of different eigenfunctions of energy:

$$u'(x) = \sum_{n=1}^{\infty} c_n \psi_n(x). \quad (19)$$

From equation (17), it is clear that all of the ψ_n 's vanish at zero, whereas $u'(x)$ does not. Thus $u'(x)$ is a wavefunction that extends beyond our Hilbert space, or in other words is a particle not confined in the infinite square well, which creates a dilemma... one which is easily solved by use of $u(x)$ as the initial state of the particle. $u(x)$, by the way, is the wavefunction that would be obtained if the potential suddenly (which can be quantitatively defined) sprung up from nowhere and captured a particle formerly in $u'(x)$.

(a)

The normalization condition is:

$$\int_0^L dx |u(x)|^2 = 1. \quad (20)$$

Let

$$u(x) = C(\sin(\pi x/L) + \sin(2\pi x/L)),$$

then normalization implies:

$$C^2 \int_0^L dx (\sin^2(\pi x/L) + 2 \sin(\pi x/L) \sin(2\pi x/L) + \sin^2(2\pi x/L)) = 1$$

Note that $u(x)$ is a superposition of the first two energy eigenfunctions (given by Eq. (17)). Since eigenfunctions are orthonormal (discussed in problem (5)), the normalization condition reduces to:

$$C^2 \int_0^L dx \frac{L}{2} (|\psi_1|^2 + |\psi_2|^2)$$

from which we conclude

$$C = \sqrt{\frac{2}{L}}.$$

The same result can be obtained through explicit integration.

(b)

Since $u(x)$ is a superposition of the first two energy eigenfunctions, a measurement of the energy of the particle will yield either E_1 or E_2 (given in Eq. (18)), each with a 50% probability. A measurement of an observable will always yield an eigenvalue of the corresponding Hermitian operator. Prof. Strovink mentions that this is the first measurement. This is important, since from another postulate of quantum theory we know that after measuring the energy, the wavefunction of the particle is subsequently described by the energy eigenfunction corresponding to the eigenvalue of energy obtained in the measurement.

(c)

The expectation value of the energy is

$$\langle E \rangle = \frac{E_1 + E_2}{2}.$$

This follows from the fact that $u(x)$ is a superposition of the first two energy eigenfunctions with equal probability to be found in either state. Thus repeated measurements on identical systems will yield E_1 half the time and E_2 the other half.

(d)

There are some subtle and important points in this part of the problem. As you saw in problem (5), a particle in an energy eigenstate of a symmetric potential always has an equal probability to be found on either the left- or right-hand side of the potential. This is not true for a superposition of energy eigenstates. This is readily seen by evaluating the expectation value of x for $u(x)$:

$$\langle x \rangle = \int_0^L x |u(x)|^2 dx \quad (21)$$

$$\langle x \rangle = \frac{2}{L} \int_0^L \left[x \sin^2 \left(\frac{\pi x}{L} \right) + x \sin^2 \left(\frac{2\pi x}{L} \right) + 2x \sin \left(\frac{\pi x}{L} \right) \sin \left(\frac{2\pi x}{L} \right) \right] dx \quad (22)$$

$$\langle x \rangle = \frac{L}{2} - \frac{16L}{9\pi^2}.$$

If we include the time evolution of $u(x)$ as described by the time-dependent Schrödinger equation, the constants in front of ψ_1 and ψ_2 acquire a time-dependence:

$$u(x, t) = c_1(t)\psi_1(x) + c_2(t)\psi_2(x). \quad (23)$$

The phase between the two wavefunctions ψ_1 and ψ_2 oscillates at a frequency:

$$\omega = \frac{\Delta E}{\hbar}, \quad (24)$$

where $\Delta E = E_2 - E_1$. Since $u(x, 0)$ has maximum probability to be found on the left-hand side, when

$$\frac{\Delta E \cdot t_0}{\hbar} = n\pi$$

($n = 1, 3, 5, \dots$) the probability to be found on the right-hand side is a maximum. So

$$t_0 = \frac{n\pi\hbar}{\Delta E}.$$